J. Fluid Mech. (1986), *vol.* 166, *pp.* 325–340
Printed in Great Britain

An extension to the short-wave asymptotics of the transmission coefficient for a semi-submerged circular cylinder

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(Received 11 November 1985)

The method of matched asymptotic expansions is used to extend the short-wave asymptotics of the transmission coefficient T by the addition of the terms of order $1/N^5$, $(\log N)^2/N^6$ and $\log N/N^6$ as $N \to \infty$ (where N = wavenumber times cylinder radius). The result is the formula

$$T = \frac{2i}{\pi N^4} \exp(-2iN)\left[1 + \frac{4\log N}{\pi N} - \frac{4}{\pi N}\left(2 - \gamma - \log 2 + \frac{i\pi}{8}\right) + \frac{8(\log N)^2}{\pi^2 N^2} - \frac{8\log N}{\pi^2 N^2}\left(5 - 2\gamma - \log 4 + \frac{i\pi}{4}\right)\right] + O\left(\frac{1}{N^6}\right) \quad \text{as } N \to \infty$$

(where γ = Euler's constant). The first term above is that derived rigorously by Ursell (1961) using an integral-equation method; the second term is that added by Leppington (1973) using matched asymptotic expansions; and the next three terms are those derived in this paper. Significant agreement between numerical values of T obtained from the completed fifth-order asymptotics and those obtained using Ursell's multipole expansions is demonstrated for $8 \leqslant N \leqslant 20$ (table 2). The extensions of the perturbation expansions for the potential in the various fluid sub-domains (used in the method of matched expansions) provide some interesting cross-checks, between the solutions for potentials occurring later in the series and determined at advanced matching stages, with those for potentials occurring earlier on and determined independently at an earlier stage in the matching process. Some examples are given.

1. The mathematical model

A sinusoidal wavetrain travelling on an inviscid, incompressible ocean of great depth is incident on a semi-submerged circular cylinder whose generators are parallel to the wave crests. Surface tension and variations in atmospheric pressure are neglected and attention is confined to irrotational, time-periodic motions for which the total potential can be taken as $\mathrm{Re}\,[\varphi(x,y)\,e^{-i\omega t}]$.

Here (x, y) are rectangular Cartesian coordinates with origin O at the centre of the cylinder, Ox pointing towards the incoming wave and Oy vertically downwards into the fluid. If all motions are assumed to be of small amplitude, then the complex-valued velocity potential φ must satisfy the following spatial boundary-value problem:

$$\nabla^2\varphi = 0 \text{ in the fluid domain,}$$

$$\varphi + \epsilon\frac{\partial\varphi}{\partial y} = 0 \text{ on the mean free surface, } y = 0$$

(where $\epsilon = g/\omega^2$ and g is the acceleration due to gravity),

$$\frac{\partial \varphi}{\partial r} = 0 \text{ on the submerged part of the cylinder}$$

(where $r = (x^2+y^2)^{\frac{1}{2}}$),

$$|\nabla \varphi| \to 0 \quad \text{as } y \to \infty.$$

$$\varphi(x,y) \sim \mathrm{e}^{-y/\epsilon}\left[\exp\left(\frac{-\mathrm{i}(x-a)}{\epsilon}\right) + \tilde{R}\left(\exp\frac{\mathrm{i}(x-a)}{\epsilon}\right)\right] \quad \text{as } x \to +\infty,$$

$$\varphi(x,y) \sim \tilde{T}[\mathrm{e}^{-\mathrm{i}(x+a)/\epsilon - y/\epsilon}] \quad \text{as } x \to -\infty$$

(where $\tilde{R}$, $\tilde{T}$ are constants and a is the cylinder radius).

In these latter equations it should be noted that the amplitude of the incident wave has been normalized by suitable choice of length and time scales and that the phase factors

$$\exp \pm \left(\frac{\mathrm{i}a}{\epsilon}\right)$$

have been inserted for convenience later on. In addition, the reflection and transmission coefficients, R and T, are related to $\tilde{R}$ and $\tilde{T}$ by the equations

$$R = \tilde{R}\exp\left(-\frac{2\mathrm{i}a}{\epsilon}\right), \quad T = \tilde{T}\exp\left(-\frac{2\mathrm{i}a}{\epsilon}\right).$$

John (1950) has shown that the problem as stated has a unique solution provided that the edge conditions, $\delta_\pm(\partial\varphi/\partial\delta_\pm) \to 0$ as $\delta_\pm \to 0$ (where $\delta_\pm = [(x \mp a)^2 + y^2]^{\frac{1}{2}}$ are satisfied.

2. Description of the method of matched asymptotic expansions as applied to the transmission problem

As Leppington (1973) gives a full discussion of the method in a more general context details will be kept to a minimum here.

In the short-wave theory the wavelengths are assumed small compared to the cylinder semi-beam ($\epsilon/a \ll 1$) and the fluid domain is subdivided initially into three regions:

(i) a right inner region $\{P: E_+P \ll a\}$,
(ii) a left inner region $\{P: E_-P \ll a\}$,
(iii) an outer region $\{P: E_\pm P \gg \epsilon\}$,

where $E_\pm$ are the points where the cylinder meets the mean free surface.

In the inner regions new coordinate axes E_+X_+, E_+Y_+ and E_-X_-, E_-Y_- are taken with E_+X_+ in the same direction as Ox, E_-X_- in the opposite direction to Ox and the Y-axes pointing vertically downwards. Coordinates in these regions are rescaled (by ϵ) so that the free-surface condition is non-dimensionalized and Laplace's equation is preserved. Thus

$$X_+ = \frac{x-a}{\epsilon}, \quad Y_+ = \frac{y}{\epsilon}, \quad X_- = \frac{-(x+a)}{\epsilon}, \quad Y_- = \frac{y}{\epsilon}.$$

Consequently $\delta_\pm = \epsilon R_\pm$ where $R_\pm = (X_\pm^2 + Y_\pm^2)^{\frac{1}{2}}$.

[The notation $\varphi(\epsilon X_+ + a, \epsilon Y_+) = \Phi_+(X_+, Y_+; \epsilon)$ and $\varphi(-\epsilon X_- - a, \epsilon Y_-) = \Phi_-(X_-, Y_-; \epsilon)$ will also be used.]

The equation of the cylinder in both systems is

$$X = \frac{f(\epsilon Y)}{\epsilon} \quad \text{where} \quad f(y) = (a^2 - y^2)^{\frac{1}{2}} - a,$$

so that the radius, as measured in the inner regions, is a/ϵ, the curvature is ϵ/a ($\ll 1$) and the cylindrical surface is indistinguishable from the vertical tangents at $E_\pm$ to lowest order.

This physically intuitive idea is modelled mathematically by rewriting the normal velocity condition on the cylinder in Cartesian form

$$\frac{\partial \Phi}{\partial X} - \frac{\mathrm{d}X}{\mathrm{d}Y}\frac{\partial \Phi}{\partial Y} = 0,$$

(where (X, Y) is either (X_+, Y_+) or (X_-, Y_-) and Φ is either Φ_+ or Φ_- depending on which inner region is under consideration), then expanding Φ_X, Φ_Y in Taylor series round $X = 0$. This leads to the condition

$$\sum_{r=0}^{\infty} g_r(\epsilon, Y)\left[\frac{\partial^{r+1}}{\partial X^{r+1}}\Phi(0, Y; \epsilon) - f'(\epsilon Y)\frac{\partial^{r+1}}{\partial X^r \partial Y}\Phi(0, Y; \epsilon)\right] = 0,$$

in which

$$g_r(\epsilon; Y) = \frac{1}{r!}\left[\frac{f(\epsilon Y)}{\epsilon}\right]^r = O(\epsilon^r), \quad f'(\epsilon Y) = O(\epsilon) \quad \text{as } \epsilon \to 0.$$

Expansion of $g_r(\epsilon, Y)$ and $f'(\epsilon Y)$ in powers of ϵ followed by substitution of any perturbation series

$$\Phi(X, Y; \epsilon) = \sum_{k \geqslant 0} a_k(\epsilon)\,\Phi_k(X, Y) \quad [a_{k+1}(\epsilon) = o(a_k(\epsilon)) \quad \text{as } \epsilon \to 0]$$

leads, after equating coefficients of various gauge factors to zero, to conditions of the form

$$\Phi_{kX}(0, Y) = V_k(Y),$$

where $V_k(Y)$ is either zero or a function of previously occurring potentials. In particular,

$$\Phi_{0X}(0, Y) = 0.$$

In addition the harmonic potential coefficients Φ_k satisfy the free-surface and edge conditions, i.e. $\Phi + \Phi_Y = 0$ on $Y = 0$ and $R(\partial \Phi / \partial R) \to 0$ as $R \to 0$. They are thus solutions of the 'classical wavemaker' type of problem but without a boundedness condition at ∞ (see Appendix A). This is replaced by a condition matching the inner solutions to the outer solutions as described later.

The outer region is divided into two further sub-regions, a boundary layer of width of order ϵ in which wave effects are noticeable, and a region, many wavelengths from the free surface in which wave effects do not appear. The lowest-order form for the outer potential in this latter region is found by setting $\epsilon = 0$ in the original problem. This leads to a homogeneous problem for which a unique solution is obtained by matching with the wave-free part of a right inner solution of appropriate order.

When a perturbation series

$$\varphi(x, y) = \sum_{k \geqslant 0} b_k(\epsilon)\,\varphi_k(x, y) \quad (b_{k+1}(\epsilon) = o(b_k(\epsilon)) \quad \text{as } \epsilon \to 0)$$

has been developed, formal substitution in the surface condition $\varphi + \epsilon\varphi_y = 0$ and equating of the coefficients of gauge factors of various orders to zero leads to a series of conditions of the form $\varphi_k = f$, where f is either zero or a function of previously determined terms. The potential coefficients must also be harmonic, satisfy $\partial\varphi/\partial r = 0$ on the submerged part of the cylinder and tend to zero as the distance from the free surface increases. In addition, matching imposes a specified behaviour at E_+. In the cases where $f \neq 0$, a particular solution can be found by conformal mapping or Green-function methods. To this may be added eigensolutions, i.e. solutions of the homogeneous problem. These are linear combinations of functions $e_m(z)$ $(m \in Z)$ where

$$e_m(z) = \mathrm{Re}\left[\mathrm{i}\left(\frac{z+a}{z-a}\right)^{2m+1}\right],$$

the coefficients in the linear combination being determined by matching with the wave-free part of the right inner solution. It may be noted that eigensolutions with $m < 0$ cannot occur since they would lead to terms in the left inner expansion (through matching) which would not satisfy the edge condition. Once the wave-free part of the outer perturbation series has been developed, matching with the inner regions is continued through the boundary layer up to the free surface by adding to the wave-free outer solution the wave terms from the inner regions expressed in outer coordinates.

It is found that the scale factors in the perturbation series occurring in the various fluid sub-domains already mentioned are all of the form $\epsilon^s(\log\epsilon)^t$ where s, t are integers $\geqslant 0$. Hence, to respect condition (iii) of theorem 1 in Fraenkel (1969, p. 223), it is necessary to adopt the matching principle proposed by Crighton & Leppington (1973) in which, for fixed s, all terms with scalings of this form (for various t) must be determined and grouped together before detailed matching takes place. When this has been achieved, the wave-free parts of the right or left inner expansions, $\Phi_\pm(X_\pm, Y_\pm; \epsilon)$ up to terms of orders ϵ^s will be denoted by $\Phi_\pm^{(s)}$. If the outer limiting process ($\epsilon \to 0$ with (x, y) fixed) is applied to this inner expansion, the result will be equivalent to that obtained by letting $R_\pm \to \infty$ in the potential coefficients (since $R_\pm = \delta_\pm/\epsilon$). When the asymptotics of these potentials have been obtained to a certain order, the result of replacing $R_\pm$ by $\delta_\pm/\epsilon$ and truncating the resulting series after terms of order ϵ^r will be denoted by $\Phi_\pm^{(s,r)}$. Similarly, let $\varphi^{(r)}$ denote the outer expansion up to terms of order ϵ^r. Application of the inner limiting process ($\epsilon \to 0$ with $(X_\pm, Y_\pm)$ fixed) is equivalent to letting $\delta_\pm \to 0$ in the potential coefficients (i.e. the φ_k). When the asymptotics of these potentials have been obtained up to a certain order, the result of replacing $\delta_\pm$ by $\epsilon R_\pm$ and truncating the resulting series after terms of order ϵ^s will be denoted by $\varphi^{(r,s)}$. The matching principle states that $\Phi_\pm^{(s,r)} = \varphi^{(r,s)}$ for any suitable r, s at our disposal. In practice, determination of the eigensolutions occurring in the various problems requires that r be taken equal to $s+1$ in the case of matching involving Φ_+, and that s be taken equal to $r+1$ in the case of matching involving Φ_- (see also Alker 1975).

3. The perturbation expansions in the various regions

The various matching cycles (right inner $\to$ outer $\to$ left inner) lead to the developments for the perturbation expansions and reflection and transmission coefficients summarized in table 1. It should be noted that the gauge factor $s(\epsilon)$ which may appear in the perturbation series for the left inner potential lies asymptotically between ϵ^5 and ϵ^6 as $\epsilon \to 0$ but is to be taken as being of different order from either $\epsilon^6(\log\epsilon)^2$ or $\epsilon^6(\log\epsilon)$.

Matching cycle	1	2	3	4
R.I.	$\Phi_+ \sim \Phi_0 + \epsilon\Phi_1$	$+\epsilon^2\Phi_2$	$+\epsilon^3\Phi_3$	Detailed
Matching	$\varphi^{(2,1)} = \Phi_+^{(1,2)}$	$\varphi^{(3,2)} = \Phi_+^{(2,3)}$	$\varphi^{(4,3)} = \Phi_+^{(3,4)}$	matching
O	$\varphi \sim \epsilon^2\varphi_0$	$+\epsilon^3 \log \epsilon\varphi_1 + \epsilon^3\varphi_2$	$+\epsilon^4 (\log \epsilon)^2\varphi_3 + \epsilon^4 \log \epsilon\varphi_4 + \epsilon^4\varphi_5$	not
Matching	$\Phi_-^{(3,2)} = \varphi^{(2,3)}$	$\Phi_-^{(4,3)} = \varphi^{(3,4)}$	$\Phi_-^{(5,4)} = \varphi^{(4,5)}$	required
L.I.	$\Phi_- \sim \epsilon^3\Psi_0$	$+\epsilon^4 \log \epsilon\Psi_1 + \epsilon^4\Psi_2$	$+\epsilon^5(\log \epsilon)^2\Psi_3 + \epsilon^5 \log \epsilon\Psi_4 + \epsilon^5\Psi_5$	$+s(\epsilon)\Psi_s + \epsilon^6(\log \epsilon)^2\Psi_6 + \epsilon^6 \log \epsilon\Psi_7 + \epsilon^6\Psi_8$
$R\sim$	$\exp(-2iN)\Big[1 - \frac{i}{2N}$	$+\left(\frac{2i}{3\pi} - \frac{1}{8}\right)\Big/N^2$	$+O\left(\frac{1}{N^3}\right)\Big]$	
$T\sim$	0	$+\frac{2i}{\pi N^4}\exp(-2iN)\Big[1 +$	$\frac{4 \log N}{\pi N} - \frac{4}{\pi N}\left(2 - \gamma - \log 2 + \frac{i\pi}{8}\right) +$	$\frac{8(\log N)^2}{\pi^2 N^2} - \frac{8 \log N}{\pi^2 N^2}\left(5 - 2\gamma - \log 4 + \frac{i\pi}{4}\right) + O\left(\frac{1}{N^2}\right)\Big]$

TABLE 1. The asymptotic development of the perturbation expansions and reflection and transmission coefficients at the various matching stages. (R.I. = right-inner region, O = wave-free outer region, L.I. = left-inner region)

The asymptotic development of the perturbation series begins in the right-inner region where the lowest-order term corresponds to total reflection of the incident wave. Matching with the wave-free outer region does not, therefore, involve this term. The presence of gauge factors other than positive integer powers of ϵ in the subsequent development of Φ_+ is discounted, since the particular Φ multiplying such a gauge factor would have to be an eigensolution of the inner problem and therefore unbounded at infinity (see Appendix A). Matching with the wave-free part of the outer solution would, in consequence, dictate the presence of a term with a multipole singularity at E_- and hence (through matching again) the presence of a term in the series for Φ_- which does not satisfy the edge condition. In the same way it can be shown that the various Φ_i do not contain eigensolutions. Note, however, that these results are not obvious at the outset, but are established in a 'step-by-step' manner for the various matching cycles in turn.

As was stated in §2 the right-inner potentials are harmonic and satisfy the free-surface and edge conditions. In addition, they satisfy the following conditions on $X_+ = 0$ (the $+$ suffixes in X_+, Y_+, R_+ will now be omitted for brevity):

$$\begin{aligned}
\Phi_{0X}(0, Y) &= 0,\\
\Phi_{1X}(0, Y) &= -\frac{1}{2a}\frac{\mathrm{d}}{\mathrm{d}Y}\left(Y^2\frac{\mathrm{d}}{\mathrm{d}Y}\right)\Phi_0(0, Y),\\
\Phi_{2X}(0, Y) &= -\frac{1}{2a}\frac{\mathrm{d}}{\mathrm{d}Y}\left(Y^2\frac{\mathrm{d}}{\mathrm{d}Y}\right)\Phi_1(0, Y),\\
\Phi_{3X}(0, Y) &= -\frac{1}{2a}\frac{\mathrm{d}}{\mathrm{d}Y}\left(Y^2\frac{\mathrm{d}}{\mathrm{d}Y}\right)\Phi_2(0, Y)+\frac{1}{8a^2}\frac{\mathrm{d}}{\mathrm{d}Y}\left(Y^4\frac{\mathrm{d}}{\mathrm{d}Y}\right)\Phi_{1X}(0, Y)\\
&\quad -\frac{1}{8a^3}\frac{\mathrm{d}}{\mathrm{d}Y}\left(Y^4\frac{\mathrm{d}}{\mathrm{d}Y}\right)\Phi_0(0, Y)+\frac{1}{48a^3}\frac{\mathrm{d}}{\mathrm{d}Y}\left(Y^6\frac{\mathrm{d}}{\mathrm{d}Y}\right)\Phi_0(0, Y).
\end{aligned}\tag{1}$$

Φ_0 is an eigensolution with an incoming wave, so must be the standing wave solution $2\exp(-Y)\cos X$. The subsequent velocity distributions on $X = 0$ are respectively of orders $\exp(-Y)$, $1/Y^2$ and $1/Y$ as $Y\to\infty$. Havelock's (1929) integral solution (Appendix A) can, therefore, be applied to determine Φ_1, Φ_2 and Φ_3 since the velocity distributions decay sufficiently rapidly to ensure convergence of the integrals involved. Nevertheless, the decay of order $1/Y$ is not of sufficient strength to produce only vanishingly small terms in the far-field form of Φ_3, which contains also a term which is $O(1)$ as $R\to\infty$ (the vortex term in (3)). At the matching stage $\varphi^{(4,3)} = \Phi_+^{(3,4)}$, this term is therefore added on to the required asymptotic form of φ_2 (see table 1), at E_+. No contradiction is involved. The solution for φ_2, which was determined completely at the previous matching stage, $\varphi^{(3,2)} = \Phi_+^{(2,3)}$, does contain an exactly corresponding term in its asymptotic form near E_+ (see comment after (5)). This term is of higher order than demanded for the asymptotic form by the matching $\varphi^{(3,2)} = \Phi_+^{(2,3)}$ but appears nevertheless because of other conditions imposed on φ_2, and provides useful evidence of the cohesion of the matching principle.

The asymptotic forms of the Φ_i in the far field, required for matching purposes, are now summarized below. Here $X = R\cos\theta$, $Y = R\sin\theta$, and the asymptotic results hold for $R\to\infty$ and $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. The wave parts are also included in the cases where they have been derived.

$$\Phi_0(X, Y) = 2\exp(-Y)\cos X$$

$$\Phi_1(X, Y) = -\frac{\mathrm{i}}{2a}\exp(\mathrm{i}X-Y)+\frac{4}{\pi a}\left(\frac{\sin\theta}{R}-\frac{\cos 2\theta}{R^2}-\frac{8\sin 3\theta}{R^3}\right)+O\left(\frac{1}{R^4}\right)$$

$$\begin{aligned}\Phi_2(X, Y) = {} & \frac{1}{a^2}\left(\frac{2\mathrm{i}}{3\pi}-\frac{1}{8}\right)\exp(\mathrm{i}X-Y)-\frac{8}{\pi^2a^2R}(\theta\cos\theta-\sin\theta\log R)\\ & -\frac{8\sin\theta}{\pi^2a^2R}\left(2-\gamma+\frac{\mathrm{i}\pi}{8}\right)-\frac{8}{\pi^2a^2R^2}(\theta\sin 2\theta+\cos 2\theta\log R)\\ & +\frac{8\cos 2\theta}{\pi^2a^2R^2}\left(3-\gamma+\frac{\mathrm{i}\pi}{8}\right)-\frac{44\sin 2\theta}{\pi a^2R^2}+O\left(\frac{\log R}{R^3}\right)\end{aligned} \quad (2)$$

$$\begin{aligned}\Phi_3(X, Y) = {} & -\frac{4\theta}{\pi^2a^3}+\frac{8\sin\theta}{\pi^3a^3R}\left[(\log R)^2-2\log R\left(3-\gamma+\frac{\mathrm{i}\pi}{8}\right)\right]\\ & -\frac{16\theta\cos\theta\log R}{\pi^3a^3R}+O\left(\frac{1}{R}\right).\end{aligned} \quad (3)$$

The result for Φ_1 is derived essentially by use of Watson's Lemma, while the integrals occurring in Φ_2 can ultimately be reduced to the double-integral form covered by the theorem in Appendix B. The leading far-field terms stated for Φ_3 depend only on the form of $\Phi_{3X}(0, Y)$ far down the 'wavemaker' ($X = 0$) since a finite portion of the wavemaker contributes dipole and higher-order terms in the far field. The same applies to terms which are of negative exponential order far down the 'wavemaker', so that examination of the equation for $\Phi_{3X}(0, Y)$, (1), shows that the leading far-field terms in Φ_3 will be generated by the term $-(1/2a)(\mathrm{d}/\mathrm{d}Y(Y^2(\mathrm{d}/\mathrm{d}Y))\Phi_2(0, Y)$. Robertson (1984, Appendix C 3) has shown that the form this takes as $Y\to\infty$ can be obtained by differentiation of the asymptotics of $\Phi_2(0, Y)$ (equation 2 with $\theta = \frac{1}{2}\pi$). This leads to the result

$$\Phi_{3X}(0, Y) = \frac{4}{\pi^2a^3}\left[\frac{1}{Y}-\frac{2\log Y}{Y^2}+\frac{9-2\gamma+\mathrm{i}\pi/4}{Y^2}\right]+O\left(\frac{\log Y}{Y^3}\right) \quad \text{as } Y\to\infty$$

whence Robertson (1984, Appendix C 4) derives the leading terms given in the far-field form of Φ_3.

Attention is now turned to the solutions for the φ_i occurring in the outer perturbation series. It is first recalled that these are, of course, harmonic, satisfy the zero-normal-velocity condition on the submerged part of the cylinder and tend to zero at infinity. Their definitions are completed by specification of their behaviour near E_+ (as determined by the matching principle) and on the free surface. These specifications and the solutions for the φ_i are now detailed. First note, however, that in what follows $x = a+\delta_+\cos\theta$, $y = \delta_+\sin\theta$, and $z = x+\mathrm{i}y$; further, any logarithms appearing are defined with a cut from a to $-\infty$, and the asymptotic relations apply when $\delta_+\to 0$; again the $+$ suffix will be dropped for brevity.

(i) $\varphi_0(x,0) = 0,\quad \varphi_0(x,y) \sim \dfrac{4\sin\theta}{\pi a\delta},\quad \varphi_0(x,y) = \dfrac{2}{\pi a^2}\mathrm{Re}\left[\dfrac{\mathrm{i}(z+a)}{(z-a)}\right];$

(ii) $\varphi_1(x,0) = 0,\quad \varphi_1 \sim -\dfrac{8\sin\theta}{\pi^2a^2\delta},\quad \varphi_1 = -\dfrac{2}{\pi a}\varphi_0;$ (4)

(iii) $\varphi_2(x,0) = -\varphi_{0y}(x,0), \quad \varphi_2 \sim -\dfrac{4\cos 2\theta}{\pi a\delta^2} - \dfrac{8}{\pi^2 a^2}\left(\dfrac{\theta\cos\theta - \sin\theta\log\delta}{\delta}\right)$

$$-\frac{8}{\pi^2 a^2}\left(2-\gamma+\frac{\mathrm{i}\pi}{8}\right)\frac{\sin\theta}{\delta},$$

$$\varphi_2(x,y) = -\frac{4}{\pi a}\operatorname{Re}\left[\frac{1}{(z-a)^2}+\frac{\mathrm{i}}{\pi a^2}\left(\frac{z+a}{z-a}\right)\log\left(\frac{z+a}{z-a}\right)\right]$$

$$-\frac{2}{\pi a}\left(2-\gamma-\log 2a+\frac{\mathrm{i}\pi}{8}\right)\varphi_0. \quad (5)$$

(The vortex term matching with that in Φ_3 comes from the right-edge asymptotics of the second term above.)

(iv) $\varphi_3(x,0) = 0, \quad \varphi_3(x,y) \sim \dfrac{8\sin\theta}{\pi^3 a^3\delta}, \quad \varphi_3 = \dfrac{2}{\pi^2 a^2}\varphi_0;$

(v) $\varphi_4(x,0) = -\varphi_{1y}(x,0) = -\dfrac{2}{\pi a}\varphi_2(x,0),$

$$\varphi_4 \sim -\frac{2}{\pi a}(\text{asymptotic form of } \varphi_2) - \frac{2}{\pi a}(\text{asymptotic form of } \varphi_1),$$

$$\varphi_4 = -\frac{2}{\pi a}(\varphi_2+\varphi_1);$$

(vi) $\varphi_5(x,0) = -\varphi_{2y}(x,0),$

$$\varphi_5 = -\frac{32\sin 3\theta}{\pi a\delta^3} - \frac{8}{\pi^2 a^2\delta^2}(\theta\sin 2\theta+\log\delta\cos 2\theta)$$

$$+\frac{8\cos 2\theta}{\pi^2 a^2\delta^2}\left(3-\gamma+\frac{\mathrm{i}\pi}{8}\right) - \frac{44\sin 2\theta}{\pi a^2\delta^2}$$

$$+\frac{8\sin\theta}{\pi^3 a^3\delta}\left[(\log\delta)^2 - 2\log\delta\left(3-\gamma+\frac{\mathrm{i}\pi}{8}\right)\right]$$

$$-\frac{16\theta\cos\theta\log\delta}{\pi^3 a^3\delta} + O\left(\frac{1}{\delta}\right).$$

It follows that

$$\varphi_5(x,y) = \operatorname{Re}[\omega(z)] - \frac{2}{\pi a}\left(3-\gamma-\log 2a+\frac{\mathrm{i}\pi}{8}\right)\varphi_2 + C\varphi_0(x,y),$$

where

$$\omega(z) = -\frac{4\mathrm{i}}{\pi a^4}\left(\frac{z+a}{z-a}\right)^3 + \frac{4\mathrm{i}}{\pi a^2(z-a)^2} + \frac{8}{\pi^2 a^2}\log\left(\frac{z+a}{z-a}\right)/(z-a)^2$$

$$+\frac{4\mathrm{i}}{\pi^3 a^4}\left(\frac{z+a}{z-a}\right)\left[\log\left(\frac{z+a}{z-a}\right)\right]^2$$

and C is a constant which can only be determined by finding the $O(1/R)$ term in the far-field form of Φ_3. This has not been attempted here.

The asymptotic forms of the φ_i near E_- are now required for the purpose of matching with the left inner region. To this end a new angle θ is introduced equal to $\pi - \arg(z+a)$ and z in the solutions for the φ_i above is replaced by $-a-\delta_-\,\mathrm{e}^{-\mathrm{i}\theta}$.

The results are detailed below and the $-$ suffix is dropped for brevity. The order terms apply for $\delta_- \to 0$.

$$\varphi_0 = \frac{\delta \sin\theta}{\pi a^3} - \frac{\delta^2 \sin 2\theta}{2\pi a^4} + \frac{\delta^3 \sin 3\theta}{4\pi a^5} + O(\delta^4),$$

$$\varphi_1 = -\frac{2}{\pi a} \text{ (asymptotic form of } \varphi_0\text{)},$$

$$\varphi_2 = -\frac{1}{\pi a^3} + \frac{\delta \cos\theta}{\pi a^4}$$
$$-\frac{2}{\pi^2 a^4}\left[\delta(\sin\theta \log\delta + \theta\cos\theta) + \left(2 - \gamma - 2\log 2a + \frac{i\pi}{8}\right)\delta \sin\theta\right]$$
$$+\frac{\delta^2 \sin 2\theta \log\delta}{\pi^2 a^5} + O(\delta^2),$$

$$\varphi_3 = \frac{2\delta \sin\theta}{\pi^3 a^5} + O(\delta^2),$$

$$\varphi_4 = \frac{2}{\pi^2 a^4} + \frac{4\delta}{\pi^3 a^5}(\sin\theta \log\delta + \theta\cos\theta) - \frac{2\delta\cos\theta}{\pi^2 a^5}$$
$$+\frac{4}{\pi^3 a^5}\delta \sin\theta\left(3 - \gamma - 2\log 2a + \frac{i\pi}{8}\right) + O(\delta^2 \log\delta)$$

$$\varphi_5 = \frac{2\log\delta}{\pi^2 a^4} + \frac{2}{\pi^2 a^4}\left(3 - \gamma - 2\log 2a + \frac{i\pi}{8}\right) + \frac{2\sin\theta\,\delta(\log\delta)^2}{\pi^3 a^5}$$
$$-\frac{2\delta\log\delta}{\pi^3 a^5}\left[\left(\pi - 2\theta)\cos\theta - 2\sin\theta(3 - \gamma - 2\log 2a + \frac{i\pi}{8}\right)\right] + O(\delta).$$

The potentials Ψ_i $(0 \leqslant i \leqslant 4)$ can now be determined explicitly (by inspection or using Lewy's 1946 reduction method) from a knowledge of the values of their normal derivatives on $X_- = 0$ and the leading terms of their asymptotic forms as $R_- \to \infty$, as determined by matching. In addition, the wave-parts of Ψ_5, Ψ_6, Ψ_7 (W_5, W_6, W_7 say), can be found without detailed knowledge of their asymptotics, while Ψ_8 is an eigensolution with no incoming wave and is therefore wave-free (Appendix A). The relevant data and solutions, where appropriate, are listed below (the $-$ suffix being dropped for brevity). Several examples of the internal consistency of the matching process occur here and will be noted as they arise.

$$\text{(i)}\quad \Psi_{0X}(0, Y) = 0, \quad \Psi_0(X, Y) \sim \frac{R\sin\theta}{\pi a^3}, \quad \Psi_0(X, Y) = \frac{Y-1}{\pi a^3}.$$

The asymptotic form and solution for Ψ_0 are determined during the first matching cycle at the matching stage $\Phi_-^{(3,2)} = \varphi^{(2,3)}$ (table 1). During the next cycle at the stage $\Phi_-^{(4,3)} = \varphi^{(3,4)}$ the term $-1/\pi a^3$ (from the left edge asymptotics of φ_2) is added on to the required asymptotic form of Ψ_0, which thus becomes the exact solution.

$$\text{(ii)}\quad \Psi_{1X}(0, Y) = 0, \quad \Psi_1(X, Y) \sim \frac{-4R\sin\theta}{\pi^2 a^4}, \quad \Psi_1 = \frac{-4\Psi_0}{\pi a} = \frac{-4(Y-1)}{\pi^2 a^4}.$$

The asymptotic form and solution for Ψ_1 are determined during the second matching cycle at the matching stage $\Phi_-^{(4,3)} = \varphi^{(3,4)}$. During the next cycle at the stage

$\Phi_-^{(5,4)} = \varphi^{(4,5)}$, a term $2/\pi^2 a^4$, from the left edge asymptotics of φ_4, and an equal term arising from the term $2\log\delta/\pi^2 a^4$ in the left edge asymptotics of φ_5, are added to the required asymptotic form of Ψ_1. It is seen, therefore, that the required asymptotic form at this later stage is the exact solution.

(iii) $$\Psi_{2X}(0, Y) = -\frac{1}{2a}\frac{\mathrm{d}}{\mathrm{d}Y}\left(Y^2\frac{\mathrm{d}}{\mathrm{d}Y}\right)\Psi_0(0, Y) = \frac{-Y}{\pi a^4},$$

$$\Psi_2 \sim \frac{2R\cos\theta - R^2\sin 2\theta}{2\pi a^4} - \frac{2R(\theta\cos\theta + \sin\theta\log R)}{\pi^2 a^4} - \frac{2R\sin\theta(2-\gamma-2\log 2a+\mathrm{i}\pi/8)}{\pi^2 a^4}.$$

Thus

$$\Psi_2 = \frac{2R\cos\theta - R^2\sin 2\theta}{2\pi a^4} - \frac{2(R\sin\theta\log R + R\theta\cos\theta - 1 - \log R)}{\pi^2 a^4}$$
$$\frac{-2(2-\gamma-2\log 2a+\mathrm{i}\pi/8)(Y-1)}{\pi^2 a^4} + \frac{2}{\pi^2 a^4}\,\mathrm{Re}\,[\mathrm{e}^{\mathrm{i}z}E_1(\mathrm{i}z)] + \frac{2\mathrm{i}\exp(\mathrm{i}X - Y)}{\pi a^4}$$

where $$E_1(\omega) = \int_\omega^\infty \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t.$$

The asymptotic form and solution for Ψ_2 are determined also during the second matching cycle at the matching stage $\Phi_-^{(4,3)} = \varphi^{(3,4)}$. During the next cycle, at the stage $\Phi_-^{(5,4)} = \varphi^{(4,5)}$, the two leading terms in the left-edge asymptotics of φ_5 cause the required asymptotic form of Ψ_2 to be modified by the addition of the terms $2\log R/\pi^2 a^4$ and $2(3-\gamma-2\log 2a+\mathrm{i}\pi/8)/\pi^2 a^4$. No contradiction is involved. The asymptotic form required at the later matching stage is seen to consist of the first three terms in the solution.

(iv) $$\Psi_{3X}(0, Y) = 0, \quad \Psi_3 \sim \frac{8R\sin\theta}{\pi^3 a^5}, \quad \Psi_3 = \frac{8(Y-1)}{\pi^3 a^5} = \frac{8\Psi_0}{\pi^2 a^2}.$$

(v) $$\Psi_{4X}(0, Y) = -\frac{1}{2a}\frac{\mathrm{d}}{\mathrm{d}Y}\left(Y^2\frac{\mathrm{d}}{\mathrm{d}Y}\right)\Psi_1(0, Y) = -\frac{4}{\pi a}\Psi_{2X}(0, Y)$$

$$\Psi_4 \sim -\frac{4}{\pi a}(\text{asymptotic form of } \Psi_2) - \frac{2}{\pi a}(\text{asymptotic form of } \Psi_1).$$

Thus $$\Psi_4 = -\frac{4}{\pi a}\Psi_2 - \frac{2}{\pi a}\Psi_1.$$

(vi) $$\Psi_{5X}(0, Y) = -\frac{1}{2a}\frac{\mathrm{d}}{\mathrm{d}Y}\left(Y^2\frac{\mathrm{d}}{\mathrm{d}Y}\right)\Psi_2(0, Y),$$

whence

$$\Psi_{5X}(0, Y) = \frac{2}{\pi^2 a^5}\left[Y\log Y + \left(3-\gamma-2\log 2a+\frac{\mathrm{i}\pi}{8}\right)Y\right] + \frac{\mathrm{i}}{\pi a^5}\frac{\mathrm{d}}{\mathrm{d}Y}(Y^2\mathrm{e}^{-Y}) + \frac{1}{\pi^2 a^5}Ei(Y)\frac{\mathrm{d}}{\mathrm{d}Y}(Y^2\mathrm{e}^{-Y})$$

where $$Ei(Y) = \fint_{-Y}^\infty \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t.$$

The wave part of $\Psi_5(W_5)$ can be found by a combination of Lewy's reduction method and the Havelock wavemaker solution. It is given by

$$W_5(X, Y) = -\frac{8\mathrm{i}}{\pi^2 a^5}\left(2-\gamma-\log 2a+\frac{\mathrm{i}\pi}{8}\right)\exp(\mathrm{i}X-Y).$$

(vii) $$\Psi_{6X}(0, Y) = -\frac{1}{2a}\frac{\mathrm{d}}{\mathrm{d}Y}\left(Y^2\frac{\mathrm{d}}{\mathrm{d}Y}\right)\Psi_3(0, Y) = \frac{8}{\pi^2 a^2}\Psi_{2X}(0, Y).$$

Hence $$W_6(X, Y) = \frac{8}{\pi^2 a^2}W_2(X, Y) = \frac{16\mathrm{i}}{\pi^3 a^6}\exp(\mathrm{i}X-Y).$$

(viii) $$\Psi_{7X}(0, Y) = -\frac{1}{2a}\frac{\mathrm{d}}{\mathrm{d}Y}\left(Y^2\frac{\mathrm{d}}{\mathrm{d}Y}\right)\Psi_4(0, Y)$$

$$= -\frac{4}{\pi a}\Psi_{5X}(0, Y)-\frac{2}{\pi a}\Psi_{4X}(0, Y).$$

(Recall that $\Psi_4 = -4/\pi a\Psi_2 - 2/\pi a\Psi_1$ and that Ψ_2 generates waves in Ψ_5 while Ψ_1 generates waves in Ψ_4.)

Thus $$W_7(X, Y) = -\frac{4}{\pi a}W_5(X, Y)-\frac{2}{\pi a}W_4(X, Y)$$

$$= \frac{16\mathrm{i}}{\pi^3 a^6}\left(5-2\gamma-2\log 2a+\frac{\mathrm{i}\pi}{4}\right)\exp(\mathrm{i}X-Y).$$

(ix) Ψ_8 is an eigensolution and therefore wave-free.

When the wave terms in the left inner expansion are collected together and expressed in outer coordinates it is seen that

$$\tilde{T} = \frac{2\mathrm{i}}{\pi}\left(\frac{\epsilon}{a}\right)^4\left[1-\frac{4}{\pi}\frac{\epsilon}{a}\log\epsilon-\frac{4\epsilon}{\pi a}\left(2-\gamma-\log 2a+\frac{\mathrm{i}\pi}{8}\right)+\frac{8}{\pi^2}\frac{\epsilon^2}{a^2}(\log\epsilon)^2\right.$$
$$\left.+\frac{8}{\pi^2}\frac{\epsilon^2\log\epsilon}{a^2}\left(5-2\gamma-2\log 2a+\frac{\mathrm{i}\pi}{4}\right)+O\left(\frac{\epsilon^2}{a^2}\right)\right].$$

Hence (with $N = a/\epsilon$) the transmission coefficient T is given by

$$T = \frac{2\mathrm{i}}{\pi N^4}\exp(-2\mathrm{i}N)\left[1+\frac{4\log N}{\pi N}-\frac{4}{\pi N}\left(2-\gamma-\log 2+\frac{\mathrm{i}\pi}{8}\right)+\frac{8(\log N)^2}{\pi^2 N^2}\right.$$
$$\left.-\frac{8\log N}{\pi^2 N^2}\left(5-2\gamma-\log 4+\frac{\mathrm{i}\pi}{4}\right)\right]+O\left(\frac{1}{N^6}\right)$$

when appeal is made to the dependence of T on the ratio ϵ/a only.

4. Comparison of the values of T given by the completed fifth-order asymptotics with those obtained using Ursell's multipole expansions, for $8 \leqslant N \leqslant 20$

The multipole expansion method of Ursell (1949) represents the scattered potential by a superposition of multipole singularities placed at the centre of the circular cylinder. These singularities are either symmetric or antisymmetric in x, so it is convenient to consider symmetric and antisymmetric parts of the scattered potential

N	ER(N) $\times 10^5$	Re($T(N)$) $\times 10^5$ A	Re($T(N)$) $\times 10^5$ M	Im($T(N)$) $\times 10^5$ A	Im($T(N)$) $\times 10^5$ M	$\lvert T(N)\rvert \times 10^5$ A	$\lvert T(N)\rvert \times 10^5$ M	arg($T(N)$) A	arg($T(N)$) M
8	0.85	−6.37	−6.22	−17.8	−17.0	18.9	18.1	−1.91	−1.92
9	0.47	−8.44	−8.09	8.14	7.92	11.7	11.3	2.37	2.37
10	0.27	7.11	6.93	2.83	2.70	7.65	7.43	0.379	0.372
11	0.17	−0.244	−0.268	−5.19	−5.07	5.19	5.08	−1.62	−1.62
12	0.11	−3.24	−3.18	1.66	1.65	3.64	3.58	2.67	2.66
13	0.07	2.06	2.04	1.64	1.60	2.63	2.59	0.671	0.666
14	0.05	0.470	0.456	−1.89	−1.87	1.95	1.92	−1.33	−1.33
15	0.03	−1.44	−1.43	0.268	0.271	1.47	1.45	2.96	2.95
16	0.02	0.648	0.646	0.925	0.915	1.13	1.12	0.960	0.956
17	0.02	0.448	0.442	−0.761	−0.757	0.883	0.876	−1.04	−1.04
18	0.01	−0.696	−0.692	−0.0728	−0.070	0.699	0.695	−3.04	−3.04
19	0.01	0.179	0.179	0.532	0.529	0.561	0.558	1.25	1.24
20	0.007	0.333	0.330	−0.311	−0.310	0.455	0.453	−0.752	−0.751

TABLE 2. Values of $T(N)$ (for $8 \leqslant N \leqslant 20$) as given by the fifth-order asymptotic formula (A) and by multipole expansions (M)

separately. The parts may be written as a superposition of a source or dipole, respectively, and a sequence of wave-free potentials. The resulting series is truncated and the boundary condition on the cylinder is imposed at a finite number of appropriately chosen points. By systematically increasing the number of points and inverting the resulting set of linear equations, a sequence of approximations to the transmission coefficient is obtained.

As the wavelength is reduced two problems arise. The multipole expansions themselves converge more slowly and it becomes necessary to include more and more terms to resolve the details of the flow. Consequently, rounding errors may accumulate. More seriously, there is near cancellation of the contributions to T from the symmetric and antisymmetric parts (these contributions are individually of order 1 while T is of order N^{-4}) so that high accuracy is required in calculating these contributions. As a result, one cannot regard the multipole calculations as a check on the asymptotic calculations. The aim is rather to establish a range of N over which numerically significant agreement is obtained between the asymptotics and the multipole calculations, the latter carried out to the greatest accuracy possible with available computing facilities. In this context, double-precision arithmetic was used throughout the calculations while the expressions for the source and dipole terms (involving sine and cosine integrals and finite quadratures) were evaluated using NAG routines with a maximum absolute error of 10^{-15} (according to the specifications of the routines). The linear equations were inverted using a NAG routine which is stated to produce solutions with residuals which are zero to machine accuracy (i.e. maximum absolute error of around 10^{-18} in this case).

Calculations of T and R were actually performed for the ranges $N = 0.01(0.01)0.1$, $0.2(0.1)1.0, 1.5(0.5)5.0, 6(1)20$. The formulae used satisfied identically the relationships $|R|^2+|T|^2 = 1$, $|\arg R - \arg T| = \frac{1}{2}\pi$ (modulo π) and the results were in agreement with those obtained by Martin & Dixon (1983) using a different computational scheme (they consider values of N up to 10).

The results appropriate to this paper (values of T for $8 \leqslant N \leqslant 20$) are given in table 2 and are derived using multipole expansions of up to 80 terms. For $8 \leqslant N \leqslant 16$, the values of $|T(M;N)|$ (M = number of terms used in multipole expansion) are stable,

$M\diagdown N$	16	17	18	19	20
50	1.11595	0.87083	0.68910	0.55027	0.44416
60	1.11914	0.87472	0.69295	0.55552	0.45005
70	1.12004	0.87593	0.69446	0.55733	0.45216
76	1.12022	0.87621	0.69484	0.55781	0.45273
77	1.12024	0.87624	0.69489	0.55786	0.45280
78	1.12026	0.87627	0.69493	0.55791	0.45286
79	1.12027	0.87629	0.69496	0.55796	0.45292
80	1.12028	0.87631	0.69499	0.55800	0.45297

TABLE 3. Numerical convergence of $|T(M;N)| \times 10^5$ for $16 \leqslant N \leqslant 20$

to 3 significant figures, from $M = 50$ onwards in all cases. At the latter end of the sequences ($M \geqslant 76$) the variations in the values of $|T(M;N)|$ are occurring in the sixth significant figure and will not (on the basis of the numerical evidence) have any subsequent effect on the third figure. (An indication of the rate of convergence is given in table 3 for $N = 16$.) In the cases $17 \leqslant N \leqslant 20$, variations are still occurring in the fifth significant figure of $|T(M;N)|$ for $M \geqslant 76$, so that the third significant figure must be considered in doubt. It has been quoted in table 2 (using $|T(M;N)|$ at $M = 80$) since the values thereby given will be underestimates of the multipole limit ($T(M;N)$ is increasing with M) and this limit will, if anything, be nearer the asymptotic value than indicated by table 2. An indication of the rate of convergence of the sequences for $17 \leqslant N \leqslant 20$ is also given in table 3. Table 2 shows significant agreement between the computed values and those obtained using the fifth-order asymptotic formula

$$T \sim \frac{2\mathrm{i}}{\pi N^4} \exp(-2\mathrm{i}N)\left[1 + \frac{4 \log N}{\pi N} - \frac{4}{\pi N}\left(2 - \gamma - \log 2 + \frac{\mathrm{i}\pi}{8}\right)\right].$$

(The error estimate, $\mathrm{ER}(N)$, for this formula is the magnitude of the first neglected term $16(\log N)^2/\pi^3 N^6$). The occurrence of small relative differences of less than 1 % for the larger values of N provides strong evidence of a region of overlap.

5. Summary and conclusions

An asymptotic expansion of the transmission coefficient as $N \to \infty$ has been derived, including all the fifth-order terms and two sixth-order terms, by use of the method of matched asymptotic expansions. To achieve this has necessitated extending the perturbation series for the potential in the various fluid regions to order ϵ^3 (in the case of the right inner region), ϵ^4 (in the case of the outer region) and $\epsilon^6 \log \epsilon$ (in the case of the left inner region). (Recall that the coefficients of the various gauge factors in the right inner, outer and left inner perturbation series are denoted by Φ_i, φ_i and Ψ_i respectively.) At the matching stage at which a particular φ_i is determined, the matching principle identifies the dominant behaviour of this particular φ_i near the right edge of the cylinder as equivalent to that due to a superposition of multipole-type singularities (situated at the right edge) of orders $(\log \delta_+)^m/\delta_+{}^n$ as $\delta_+ \to 0$, where m, n are integers with $m \geqslant 0$, $n \geqslant 1$. The combination of such singular terms may be characterised as the 'principal behaviour' of the φ_i in question near the right edge. It is prescribed by the vanishingly small terms in the far-field forms of those Φ_k which have already been found and, together with the other conditions imposed at the free surface, on the cylinder and at infinity, is sufficient to determine φ_i completely. Once this has been achieved, it has been evident in some cases (specifically $i = 2, 4, 5$) that the principal behaviour near the right edge is supplemented by weaker behaviour equivalent to a superposition of terms which are either source-like, vortex-like or vanishingly small. It may be conjectured, therefore, that this weak behaviour will be prescribed (during a later matching cycle) by terms which are not vanishingly small in the far-field forms of certain Φ_k which are as yet undetermined. In other words, the 'weak' behaviour of a φ_i predicts 'strong' behaviour in the far field of one or more of the Φ_k which will appear in later matching cycles. (Such an inter-relationship has already been pointed out between φ_2 and Φ_3 in §3.) In the same way, 'weak' behaviour in a Ψ_i in the far field predicts 'strong' behaviour (near the left edge) in one or more of the φ_k which are determined during a later matching cycle. (Several examples of this useful system of cross-checks are noted in §3.) It seems, therefore, that the matching principle as propounded by Crighton & Leppington (1973) has a striking internal cohesion which, to the author's knowledge, has not been previously noted.

From a numerical viewpoint, Ursell's multipole expansion method is inefficient for large values of N. However, the calculations have been performed to as high an accuracy as possible with available computing facilities and it is believed that three significant figures have been obtained for $|T(N)|$ for $8 \leqslant N \leqslant 16$ and two significant figures for $17 \leqslant N \leqslant 20$ (see §4). Table 2 indicates a significant region of overlap with the fifth-order asymptotic formula for $8 \leqslant N \leqslant 20$. For this range of moderate values of N it is not anticipated that the addition of the sixth-order terms would improve matters. The two terms derived in this paper can be combined as

$$\frac{8 \log N}{\pi^2 N^6}\left[\log 4N + 2\gamma - 5 - \frac{i\pi}{4}\right]$$

and the expression $\log 4N + 2\gamma - 5$ is negative up to about $N = 12$, and at $N = 20$ has not advanced far enough into the asymptotic region to give meaningful results. Indeed, to obtain a value of $\log 4N$ only twice that of $5 - 2\gamma$ would necessitate taking N to be around 550. It is not anticipated that this would be computationally tractable by any method known at present.

I wish to thank Dr John Martin of Edinburgh University Mathematics Department for his help during the preparation of this paper, and Edinburgh University for the award of a Chalmers Research Scholarship. The computations were carried out on an ICL 2900 at the Edinburgh Regional Computing Centre.

Appendix A. The Classical Wavemaker family of problems

The problem (in its general form) is to find a function $U(x, y)$, harmonic in the quadrant $x > 0$, $y > 0$, and such that

$$\begin{aligned} kU + U_y &= 0 \quad (y = 0,\ x > 0), \\ U_x &= V \quad (x = 0,\ y > 0), \end{aligned}$$

where V is a prescribed function of y.

The general solution is $U(x, y) = P(x, y) + E(x, y)$ where $P(x, y)$ is a particular solution and $E(x, y)$ is a solution of the corresponding homogeneous problem. Havelock (1929) provides a particular solution in the form

$$P(x, y) = \int_0^\infty H(x, y; k; s)\ V(s)\,\mathrm{d}s,$$

where

$$H(x,y;k;s) = -2\mathrm{i}\exp[\mathrm{i}kx-k(y+s)] -\frac{2}{\pi}\int_0^\infty \frac{(u\cos uy-k\sin uy)(u\cos us-k\sin us)}{u(u^2+k^2)}\,\mathrm{e}^{-ux}\,\mathrm{d}u,$$

or

$$= -2\mathrm{i}\exp[\mathrm{i}kx-k(y+s)]+\frac{1}{2\pi}\log\left[\frac{x^2+(y-s)^2}{x^2+(y+s)^2}\right] -\frac{2}{\pi}\int_0^\infty \frac{u\cos u(y+s)-k\sin u(y+s)}{u^2+k^2}\,\mathrm{e}^{-ux}\,\mathrm{d}u.$$

When $V(y)$ is such that the integrals occurring in Havelock's solution are non-convergent (e.g. $V(y) = y$ occurs in the text) then Lewy's (1946) reduction method may often be applied.

Lewy's method shows also that $E(x,y)$ is precisely a linear combination of the functions

(i) $\mathrm{e}^{-ky}\cos kx$,

(ii) $\dfrac{kr^{2m+1}\sin(2m+1)\theta}{2m+1} - r^{2m}\cos 2m\theta \quad (m = \ldots, -2, -1, 0, 1, 2, \ldots)$,

(iii) $\mathrm{Re}\,[\mathrm{e}^{\mathrm{i}kz}E_1(\mathrm{i}kz)] - \pi\,\mathrm{e}^{-ky}\sin kx$,

where

$$E_1(\omega) = \int_\omega^\infty \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t, \quad x = r\cos\theta, \quad y = r\sin\theta.$$

The functions of the above set which satisfy the edge condition $r(\partial\varphi/\partial r)\to 0$ as $r\to 0$ are termed eigensolutions. Thus the eigensolutions are $\mathrm{e}^{-ky}\cos kx$ and those functions in (ii) for which $m \geqslant 0$. Solution (iii) has source-like behaviour at the origin implying that an eigensolution cannot supply travelling waves at ∞ (since these could only be formed by combining (i) and (iii) (cf. Alker 1975, p. 203).

Appendix B

The following theorem, proved in Robertson (1984), is required for determining the asymptotics of Φ_2 in the right inner expansion:

If f, g are two functions defined on $[0,\infty]$ such that

(*a*) $f, g \in C^\infty[0,\infty]$,

(*b*) $f^k(t)\log t, \quad g^k(t)\log t \to 0 \quad$ as $t\to\infty \quad$ for $k = 0, 1, 2\ldots$,

(*c*) $\displaystyle\int_x^\infty \frac{f^k(t)}{t}\,\mathrm{d}t$ exists for all $x > 0 \quad (k = 0, 1, 2, \ldots)$,

(*d*) $\displaystyle\int_0^\infty g^k(t)\,\mathrm{d}t$ is absolutely convergent $\quad (k = 0, 1, 2, \ldots)$,

then

$$I(z) \stackrel{D}{=} \int_0^\infty\int_0^\infty f(t)\,g(u)\,\mathrm{e}^{-ztu}\,\mathrm{d}u\,\mathrm{d}t$$

exists and

$$I(z) \sim \sum_{r=0}^\infty \frac{d_r(z)}{r!\,z^{r+1}} \quad \text{as } z\to\infty \quad \text{in } |\arg z| \leqslant \tfrac{1}{2}\pi,$$

where

$$d_r(z) = (\log z + \gamma + s_r) f^r(0)\, g^r(0) - f^r(0) \int_0^\infty g^{r+1}(t) \log \mathrm{t}\, \mathrm{d}t \\ - g^r(0) \int_0^\infty f^{r+1}(t) \log t\, \mathrm{d}t$$

and

$$s_r = \begin{cases} 0 & \text{if } r = 0, \\ \sum_{m=1}^{r} \frac{1}{m} & \text{if } r \geqslant 1. \end{cases}$$

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